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TWO-DIMENSIONAL PROCESSES OF THE UNBOUNDED UNSHOCKED COMPRESSION OF A GAS†

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The exact solution of the two-dimensional unsteady problem of the interaction of two one-dimensional non-self-similar Riemann compression waves, each of which generates an unlimited local increase in the gas density in the neighbourhood of a moving compressive piston, is constructed. Solutions are obtained in which the adiabatic exponents and the angle at which the Riemann waves interact are specially arranged to be consistent. Both limited and unlimited energy expenditure on such compression is considered. In both cases a cumulative gas jet arises in the region of Riemann wave interference, the extent to which the gas-dynamic quantities accumulate being the same as for unlimited self-similar two-dimensional compression of a gas prism. Thus it is shown that high local degrees of energy accumulation can be attained for a broad class of laws of control by unshocked compression. A phenomenon of partial gas collapse is observed.

Investigations have been carried out on one-dimensional self-similar regimes of the unlimited unshocked compression of ideal gases contained initially inside a prism, tetrahedron or conical solid [1–6]. In addition to multidimensional compression, which requires the unlimited expenditure of energy, laws of control of one-dimensional plane compression leading to an unlimited local increase in the gas density with finite energy expenditure are constructed in [6]. In that case the gas flow field can be described by a non-self-similar simple Riemann wave. Although the entire mass of the gas does not then collapse totally, it is interesting to investigate the two-dimensional interaction of two one-dimensional Riemann compression waves at a certain angle.

We will consider the problem of finding local degrees of cumulation of the gas dynamic quantities in the interference zone of these Riemann waves, and also the amount of energy expended on compression. Although for some laws of control of compression in a one-dimensional Riemann wave the energy expenditure is finite, it is not obvious that the amount of energy expended in the interaction of these waves is finite, and this requires investigation. The question of whether a regime of unshocked compression exists before unlimited density is attained, if only locally in the interference zone, also needs investigation.

The present paper discusses the solution of these problems for several classes of laws of compression control. The motion in zones of two-dimensional interaction is described using the classes of exact solutions of non-self-similar double waves, previously used to investigate the interaction of rarefaction waves [7, 8].

1. CONSTRUCTION OF A SOLUTION

Suppose that at the initial instant of time $t = 0$ an ideal gas with equation of state $p = p_0(\rho/\rho_0)^\gamma$ (p is the pressure, ρ is the density, ρ_0 and p_0 are the initial density and pressure of the gas, and γ is the adiabatic exponent) is at rest inside the dihedral angle β formed by planes R_{10} and R_{20} (Fig. 1). Planes R_{10} and R_{20} correspond to the initial position of moveable curved compressive pistons, whose plane parts R_{1t} and R_{2t}, far from the line ODS, move in accordance with the equations

$$x_2 = f_1(t), \quad \cos\beta x_2 - \sin\beta x_1 = -f_2(t) \tag{1.1}$$

It will be assumed that the pistons start moving with zero velocity $f_1'(0) = f_2'(0) = 0$ and move in the interval $t \in (0, t_*]$ with monotonely increasing velocities $f_1'(t) > 0$ and $f_2'(t) > 0$, so that the inverse functions $(f_1')^{-1} = z_1$ and $(f_2')^{-1} = z_2$ exist. We take the initial velocity of sound as the velocity scale. Then $c = c_0 = \sqrt{(\gamma p_0/\rho_0)} = 1$.

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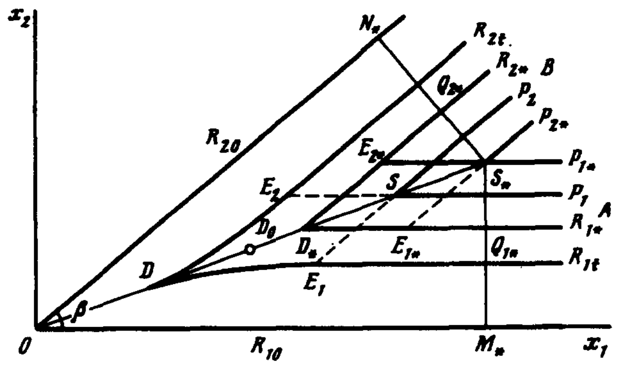

Fig. 1.

Far from the vertex of the angle in regions A and B the gas flows correspond to simple Riemann waves which can be described by the relations ($u_i(\theta)$ are the components of the velocity vector, $i = 1, 2$)

$$u_1 = 0,\quad u_2 = \theta - \frac{2}{\gamma-1},\quad \theta = \frac{2}{\gamma-1}c \quad (\text{Region } A)$$

$$u_1 = \frac{\gamma+1}{2h}\left(\theta - \frac{2}{\gamma-1}\right),\quad u_2 = -\frac{\gamma-1}{2}\left(\theta - \frac{2}{\gamma-1}\right),\quad h = \sqrt{\frac{\gamma+1}{3-\gamma}} \quad (\text{Region } B) \tag{1.2}$$

where

$$u_1'x_1 + u_2'x_2 - \left(\frac{\gamma-1}{2}\theta + u_1u_1' + u_2u_2'\right)t + \Delta_i(\theta) = 0 \tag{1.3}$$

(the prime denotes differentiation with respect to θ). The second pair of formulae (1.2) for u_1 and u_2 is written for the so-called "consistent case" [9], when

$$\operatorname{tg}\frac{\beta}{2} = \frac{1}{h}(1 < \gamma < 3) \tag{1.4}$$

The solution of the equation of double waves in the region of interference of simple waves has a very simple form here. The functions $\Delta_i(\theta)$ in (1.3) for regions A and B are determined from the given $f_i(t)$ in (1.1) and have the form [7]

$$\Delta_i(\theta) = \left(\frac{\gamma+1}{2}\theta - \frac{2}{\gamma-1}\right)y_i - f_i(y_i),\quad y_i = z_i\theta_\gamma,\quad \theta_\gamma = \theta - \frac{2}{\gamma-1} \tag{1.5}$$

To describe the gas motion in the region DE_1SE_2, we use the equations of non-self-similar double waves [8] for the function $\theta = \theta(u_1, u_2)$ and the "distribution" function $\chi(u_1, u_2)$

$$\frac{\gamma-1}{2}\theta[(1-\theta_1^2)\theta_{22} + 2\theta_1\theta_2\theta_{12} + (1-\theta_2^2)\theta_{11}] +$$
$$+\frac{\gamma-3}{2}\left(\theta_1^2 + \theta_2^2\right) + 2 = 0,\quad \theta_i = \frac{\partial\theta}{\partial u_i},\quad \theta_{ik} = \frac{\partial^2\theta}{\partial u_i \partial u_k} \tag{1.6}$$

$$(1-\theta_1^2)\chi_{22} + 2\theta_1\theta_2\chi_{12} + (1-\theta_2^2)\chi_{11} = 0,\quad \chi_{ik} = \frac{\partial^2\chi}{\partial u_i \partial u_k} \tag{1.7}$$

$$x_i = \left(u_i + \frac{\gamma-1}{2}\theta\theta_i\right)t + \chi_i,\quad i,k = 1,2 \tag{1.8}$$

Once the functions θ and χ have been found, formulae (1.8) implicitly define the gas velocity field in physical variables x_1, x_2, t.

For the consistent case (1.4), the solution of Eq. (1.6) in the region DE_1SE_2 has the form

$$\theta = \frac{2}{\gamma-1} + u_1 h + u_2 \tag{1.9}$$

and we write the general solution of Eq. (1.7) for χ in the form

$$\chi = \Psi_1(u_1) + \Psi_2(u_1 + \delta u_2), \quad \delta = \frac{\sqrt{(\gamma+1)(3-\gamma)}}{\gamma-1} \tag{1.10}$$

where Ψ_1 and Ψ_2 are arbitrary functions of their arguments. Their derivatives can be expressed in terms of given $f_i(t)$ [7] as follows:

$$\Psi_1'(\xi) = -\varkappa\{(1+h\xi)z_2(\varkappa\xi) - f_2[z_2(\varkappa\xi)]\}, \quad \varkappa = \frac{2}{(\gamma-1)\delta} \tag{1.11}$$

$$\Psi_2'(\eta) = -\frac{1}{\delta}\left\{\left(1+\frac{\gamma+1}{2}\frac{\eta}{\delta}\right)z_1\left(\frac{\eta}{\delta}\right) - f_1\left[z_1\left(\frac{\eta}{\delta}\right)\right]\right\}$$

Formulae (1.2)–(1.5) and (1.9)–(1.11) give the complete exact solution of the problem if the conditions for relations (1.3) and (1.8) to be solvable are satisfied. The velocity field is determined from these formulae.

Using (1.8) and (1.10) we can represent the Jacobian in the form

$$J = \frac{D(x_1,x_2)}{D(u_1,u_2)} = \left(\frac{2t}{3-\gamma} + \Psi_1''(\xi)\right)\left(\frac{\gamma+1}{2}t + \frac{1}{(\gamma-1)^2}\Psi_2''(\eta)\right) \tag{1.12}$$

It follows at once from (1.3), (1.5) and (1.9)–(1.11) that if the expression in one of the brackets of the Jacobian J vanishes, a gradient catastrophe occurs in one of the Riemann waves in region A or B. Thus, if a gradient catastrophe has not occurred in A and B by some particular time, it will not have occurred before then in the region of the double wave either, that is, the conditions for the implicit equations (1.8) to be solvable are satisfied.

2. THE SYMMETRIC CASE

For arbitrary laws of motion of the pistons R_{it} which lead to an unlimited increase in the gas density, considerable difficulties arise when investigating the degrees of cumulation of the gas dynamic quantities and energy expenditure. We will therefore confine ourselves below to the symmetric motion of compressive pistons, where the functions $f_i(t)$ have the form

$$f_1(t) = f_2(t) = B_1 + B_2 t + B_3(t_* - t)^\alpha, \alpha < 1 \tag{2.1}$$

$$B_1 = \frac{r_*}{1-\alpha}, \quad B_2 = -\frac{r_*\alpha}{t_*(1-\alpha)} < 0, \quad B_3 = \frac{r_*}{t_*^\alpha(1-\alpha)} < 0$$

The constants B_k are determined from the conditions

$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1(t_*) = r_*$$

where the parameters r_* and t_* determine the place and time at which the infinite velocities and densities, which increase by a power law, occur.

The laws of motion of a piston for this case were considered in [6], where it was shown that provided that

$$\frac{2\gamma}{3\gamma-1} < \alpha < 1 \tag{2.2}$$

the energy that needed to be expended on the displacement of a plane piston to reach an infinite density is finite. As $\tau \to 0$ ($\tau = t_* - t$) the degree of cumulation n of the velocity of sound ($c = O(\tau^{-n})$) on the piston is equal to $1 - \alpha$.

If $\alpha = 2/(\gamma + 1)$, $t_* = r_* = 1$, formulae (2.1) give the law of control of the compression, when in the self-similar mode a plane layer of unit thickness entirely collapses at a time $t = 1$ [10]. Certainly, this requires infinite energy loss.

In region A (1.3) becomes

$$x_2 - r_* + (t_* - t)\left(\frac{\gamma+1}{2}u_2 + 1\right) - t_*\left[\frac{\gamma-1}{2}B_2 + 1 - \right.$$
$$\left. -B_2\left(-\frac{1}{\alpha} + \frac{\gamma+1}{2}\right)\left(1 - \frac{u_2}{B_2}\right)\right]\left(1 - \frac{u_2}{B_2}\right)^{\frac{1}{\alpha-1}} = 0 \tag{2.3}$$

Using (2.3) express $\partial x_2/\partial u_2$ and $\partial^2 x_2/\partial u_2^2$ and analysing the resulting representation, we obtain a constraint on the parameters t_*, r_*, γ and α which must be satisfied for gradient catastrophe not to occur in A and B before a time $t = t_*$

$$t_* \geqslant \frac{\gamma+1}{2} r_* \alpha, \quad \alpha > \frac{2}{\gamma+1}\left(\frac{2}{\gamma+1} < \frac{2\gamma}{3\gamma-1}\right)$$

As $t \to t_*(u_2 \to \infty)$ a gradient catastrophe occurs on the piston.

In the region of the double wave we convert relations (1.8) to the form

$$\begin{aligned}
x_1 &= X_1(u_1, u_2)t + \Psi_1'(u_1) + \Psi_2'(u_1 + \delta u_2) \\
x_2 &= X_2(u_1, u_2)t + \delta\Psi_2'(u_1 + \delta u_2) \\
X_1(u_1, u_2) &= h + \frac{\gamma^2 - 2\gamma + 5}{2(3-\gamma)} u_1 + \frac{\gamma-1}{2} h u_2 \\
X_2(u_1, u_2) &= 1 + \frac{\gamma-1}{2} h u_1 + \frac{\gamma+1}{2} u_2
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&\Psi_1'(\xi) = a_1 + b_1\xi + (p_1 + q_1\xi)(1 + s_1\xi)^{\frac{1}{\alpha-1}} \\
&\Psi_2'(\eta) = a_2 + b_2\eta + (p_2 + q_2\eta)(1 + s_2\eta)^{\frac{1}{\alpha-1}} \\
&a_1 = -p_1 = -\varkappa(t_* - r_*), \quad a_2 = -p_2 = -\frac{1}{\delta}(t_* - r_*) \\
&b_1 = -\frac{2t_*}{3-\gamma}, \quad b_2 = -\frac{(\gamma-1)^2 t_*}{2(3-\gamma)} \\
&s_1 = -\frac{\varkappa}{B_2} > 0, \quad s_2 = -\frac{1}{\delta B_2} > 0 \\
&q_1 = -b_1\left(1 - \frac{2}{\alpha(\gamma+1)}\right), \quad q_2 = -b_2\left(1 - \frac{2}{\alpha(\gamma+1)}\right)
\end{aligned} \tag{2.5}$$

The lines between the double wave and the regions of simple waves E_1S and E_2S at each instant of time are straight. In fact, putting $u_1 = 0$ in (2.4), we obtain the following equation of motion of the characteristic E_1S

$$x_2 - \delta x_1 = -\frac{2}{\gamma - 1}t - \delta\Psi_1'(0)$$

In order to find the law of motion $F(x_1, x_2, t) = 0$ of a mobile curvilinear piston DE_1 for a velocity field given by (2.4), we must solve the first-order equation

$$F_{x_1}u_1 + F_{x_2}u_2 + F_t = 0 \tag{2.6}$$

This equation corresponds to the condition that the piston is impermeable, and its boundary condition defines the known law of motion of the point E_1 in a simple Riemann wave. However, the equations for characteristics (2.6), which have the form

$$\frac{dx_1}{dt} = u_1(x_1, x_2, t), \quad \frac{dx_2}{dt} = u_2(x_1, x_2, t) \tag{2.7}$$

where u_1 and u_2 are determined implicitly from (2.4), cannot be integrated in quadratures. Moreover, the law of motion of the piston DE_1 cannot be found, even in the limit (as $t \to t_*$), even though for large $|u_1|$ and $|u_2|$ the velocity field is self-similar in the neighbourhood of the cusp D.

In fact, for large ξ and η in (2.5) $\Psi_1'(\xi)$ and $\Psi_2'(\eta)$ are linear functions, formulae (2.4) asymptotically become

$$\xi_1 = \frac{x_1 - hr_*}{t - t_*} = X_1(u_1, u_2), \quad \xi_2 = \frac{x_2 - r_*}{t - t_*} = X_2(u_1, u_2) \tag{2.8}$$

where ξ_k are self-similar variables.

It is simple to integrate Eqs (2.7) with u_k from (2.8), but this does not yield the law of motion of the piston DE_1, because $u_1 = 0$ on the curve next to the simple wave in zone A and the argument about self-similarity is only justified near the cusp D.

Using relations (2.4), we can reduce the system of equations (2.7) to the form

$$\frac{du_1}{dt} = G_1(u_1, u_2, t), \quad \frac{du_2}{dt} = G_2(u_1, u_2, t) \tag{2.9}$$

where the functions G_1 and G_2 are expressed explicitly using (2.5). However, system (2.9) cannot be integrated analytically either. The domain of definition of the flow in the plane of the hodograph of velocities u_1, u_2 will depend on t. The evolution of this domain for different t is shown at a qualitative level in Fig. 2. As $t \to t_*$ the domain of definition of the double wave corresponds to the sector bounded on the left and below by the lines ON and OM, which correspond to simple Riemann waves A and B.

Of course, the formula of piston DE can be found numerically by integrating the equation of characteristics (2.7) covering the piston surface $F(x_1, x_2, t) = 0$, on the composite velocity field $t \to t_*$, determined from relations (1.2) and (1.3) for simple waves, and (2.4) for a double wave. The initial

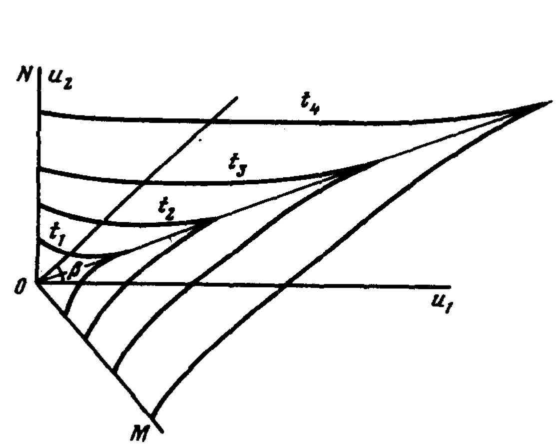

Fig. 2.

conditions for the numerical integration are determined by specifying the initial shape of the moving piston, the line R_{10}.

Despite the difficulties that are encountered in finding the shape of the piston DE analytically, the trajectory of motion of the point D and laws of variations of the velocity and density at it can be found exactly for all $t \in [0, t_*]$. Thus the local degrees of cumulation of the gas-dynamic quantities in the neighbourhood of the point D as u_1, u_2 can be found analytically.

Along the bisector OS: $x_1 = hx_2$, on which $u_1 = hu_2$, from (2.4) we obtain an ordinary differential equation from which we can find $x_1(\tau)$ $(\tau = t_* - t)$

$$x_1(\tau) = h(r_* - \tau) + h^2\tau x_1'(\tau) - h(r_* - t_* - rqx_1'(\tau))\times$$
$$\times\left(1 - 2(1-\alpha)\frac{r}{r_*}x_1'(\tau)\right)^{\frac{1}{\alpha-1}}, \quad r = \frac{t_*\varkappa}{2\alpha}, \quad q = 2 - \alpha(\gamma+1) \tag{2.10}$$

with the initial condition

$$x_1(t_*) = 0$$

It follows at once from (2.10) that $x_1'(t_*) = 0$.

Equation (2.10) is a Lagrange equation and is integrated in quadratures. Putting $-x_1'(t_*) = b(\tau)$, we can find the velocity $b(\tau) \in [0, \infty)$ at the point D from the equation

$$\frac{d\tau}{db} + h\left(1 + \frac{2}{\delta}b\right)^{-1}\tau = -h\left(1 + 2(1-\alpha)\frac{r}{r_*}b\right)^{\frac{2-\alpha}{\alpha-1}}\left(s - \frac{q\varkappa}{r_*}b\right)\left(1 + \frac{2}{\delta}b\right)^{-1} \tag{2.11}$$

$$s = 2\frac{t_*}{r_*} - \alpha(\gamma+1)$$

Integrating Eq. (2.11), we obtain a relation which implicitly determines the value of the velocity $b(\tau)$ at the point D

$$-\tau = \left(1 + \frac{2}{\delta}b\right)^{-\frac{\gamma+1}{2(\gamma-1)}}G \tag{2.12}$$

$$G = -t_* + r\int_0^b\left\{\left(s - \frac{q\varkappa}{r_*}x\right)\left(1 + 2(1-\alpha)\frac{r}{r_*}x\right)^{\frac{2-\alpha}{\alpha-1}}\left(1 + \frac{2}{\delta}x\right)^{\frac{3-\gamma}{2(\gamma-1)}}\right\}dx$$

From (2.11) we can find the degree of cumulation of the velocity b (and therefore of the velocity of sound c from (1.9)) for small τ. For $\alpha > 2/(\gamma + 1)$ in (2.11), the constant $q > 0$ and $db/dt > 0$ for all $t \in (0, t_*]$. Thus, the point D moves with a velocity $b \geqslant 0$ which increases from zero to infinity. For large b the integral in (2.12) for $\alpha > 1/h^2$ converges, that is, for $\alpha > 2/(\gamma + 1)$, $b = \infty$ it has a finite value.

If the quantity G in (2.12) does not tend to zero as $b \to \infty$, it follows from (2.12) that the asymptotic law of the increase of the velocity of the gas and the velocity of sound c as $\tau \to 0$ has the form

$$c = O\left(\tau^{-2\frac{\gamma-1}{\gamma+1}}\right) \tag{2.13}$$

Thus, the degree of cumulation of the velocity (and the velocity of sound) at the point D is independent of the quantity α and is equal to the degree of cumulation of the velocity during unlimited self-similar compression of a prism [1].

An analysing the value of the improper integral in the expression for G when $b = \infty$ is a lengthy procedure. It can be shown to be strictly less than t_* at least in some range of variation of the parameters α, γ, t_* and r_*.

3. ENERGY ESTIMATES

We will investigate the energy $E(t)$ expended in moving the pistons R_{1t} and R_{2t} up to time t_* in the case when the index α satisfies inequality (2.2). Although for plane motion the energy expenditure per unit length is finite in case (2.2), estimating the energy in the region of interference of simple waves (will it be finite?) is a non-trivial problem.

We cannot directly use an integral expression of the form

$$E(t) = 2\int_0^{t_*} \int_{DE_1(t)} P_{DE_1(t)} u_n \, dl \, dt$$

where $P_{DE_1(t)}$ is the pressure on the piston $DE_1(t)$, u_n is the normal velocity of the piston motion and dl is an element of length of arc of the piston because, as we have already noted, the law of the piston motion $DE_1(t)$ has not been found in analytic form. On the other hand, the work of the pistons R_{1t} and R_{2t} goes on increasing the internal and kinetic energy of the gas in region DE_1SE_2. Thus if we study the final state of the gas at time $t = t_*$ and the limiting shape of the piston $DE_1(t)$, we can then try to estimate the total energy $E(t_*)$ by a direct estimate of the internal energy $\varepsilon(t_*)$ and the kinetic energy $K(t_*)$.

As $t \to t_*$, from representations (2.4), Eqs (2.5) and the fact that although u_1 and u_2 both increase without limit, the estimate (2.13) is valid, we first obtain the following asymptotic forms

$$x_1 \approx x_{1f} + (p_2 + q_2\eta)(1 + s_2\eta)^{\frac{1}{\alpha-1}}, \quad x_{1f} = hr_* + (p_1 + q_1\xi)(1 + s_1\xi)^{\frac{1}{\alpha-1}} \tag{3.1}$$

$$x_2 \approx x_{2f} + \delta(p_2 + q_2\eta)(1 + s_2\eta)^{\frac{1}{\alpha-1}}, \quad x_{2f} = r_*$$

On piston DE_1 as $t \to t_*$ the quantity $\xi = u_1$ varies from zero to infinity, and $\eta = u_1 + \delta u_2$ is unbounded. Thus in the final state the shape of the piston $DE_1(t_*)$ can be represented parametrically as $x_1 = x_{1f}$, $x_2 = x_{2f}$, that is, at time $t = t_*$ the piston DE_1 becomes the rectilinear segment D_*E_{1*} (Fig. 1)

$$x_1 \in [hr_*, hr_* + p_1], \quad x_2 = r_*$$

The parallelogram $D_*E_{1*}S_*E_{2*}$ corresponds to the final region of a non-self-similar double wave.

The velocity distribution in the neighbourhood of the point D_* can be found from the asymptotic formulae which follow from (3.1) for large ξ and η

$$x_1 - hr_* \approx m_1\xi^{\frac{\alpha}{\alpha-1}} + m_2\eta^{\frac{\alpha}{\alpha-1}}, \quad x_2 - r_* \approx n_2\eta^{\frac{\alpha}{\alpha-1}} \tag{3.2}$$

$$m_1 = q_1 s_1^{\frac{1}{\alpha-1}} > 0, \quad m_2 = q_2 s_2^{\frac{1}{\alpha-1}} > 0, \quad n_2 = \delta m_2 > 0$$

It follows from (3.2) that

$$\xi = \zeta_1^{\frac{\alpha-1}{\alpha}}, \quad \eta = \zeta_2^{\frac{\alpha-1}{\alpha}}$$

$$\zeta_1 = (x_1 - \delta^{-1}x_2 - \varkappa r_*)\, m^{-1}, \quad \zeta_2 = (x_2 - r_*)\, n_2^{-1}$$

In the region $D_*E_{1*}S_*E_{2*}$ we have $\zeta_1 \geqslant 0$. From (1.9) we then obtain

$$\theta = \frac{2}{\gamma - 1} c(t_*) = \frac{2}{\gamma - 1} + \varkappa\zeta_1^{\frac{\alpha-1}{\alpha}} + \frac{1}{\delta}\zeta_2^{\frac{\alpha-1}{\alpha}}$$

Both the internal energy $\varepsilon(t)$ and the kinetic energy $K(t_*)$ for internal points of the region $D_*E_{1*}SE_{2*}$ are proportional to the double integral over that region of $c^{(2\gamma/\gamma-1)}(t_*, x_1, x_2)$. We reduce the problem of estimating the energy to an investigation of the convergence of the improper integral

$$J=\int_0^{\mu}\int_0^{\mu}\left(\alpha_1+\alpha_2\zeta_1^{\frac{\alpha-1}{\alpha}}+\alpha_3\zeta_2^{\frac{\alpha-1}{\alpha}}\right)^{\frac{2\gamma}{\gamma-1}}d\zeta_1 d\zeta_2$$

where α_k and μ are certain positive constants. The quantity J is majorized by the integral

$$J_0=(3\max\alpha_k)^{\frac{2\gamma}{\gamma-1}}\int_0^{\mu}\int_0^{\mu}(1+\zeta_1^g+\zeta_2^g)\,d\zeta_1 d\zeta_2,\quad g=\frac{2\gamma(\alpha-1)}{\alpha(\gamma-1)}$$

Since $g+1>0$ if inequality (2.2) is satisfied, we have $J_0<\infty$.

In order to estimate the total energy losses $E(t_*)$ before the final state is reached, we also need to calculate the mass balance. In fact, the final part of the original mass of gas can, in principle, collapse at time t_* to the point D_* and along lines D_*E_{1*}, D_*E_{2*}. Thus, we need to check that

$$M_0=2M_1+M_2 \tag{3.3}$$

$$M_1=\iint_{E_{1*}Q_{1*}S_*}\rho_1(x_1,x_2,t_*)\,dx_1dx_2,\quad M_2=\iint_{D_*E_{1*}S_*E_{2*}}\rho_2(x_1,x_2,t_*)\,dx_1dx_2$$

where $M_0=h\rho_0 t_*^2$ is the total mass of the compressed volume of gas, $\rho_1(x_1, x_2, t_*)$ for a simple wave is found from (1.2) and (2.3), and $\rho_2(x_1, x_2, t_*)$ for a double wave—from (1.9) and (2.4). If relation (3.3) is an identity, it follows from the relation $J_0<\infty$ that the total energy losses are finite. If M_0 is greater than the right-hand side of Eq. (3.3), during the compression process part of the mass collapses at time t_* and is concentrated along the lines D_*E_{1*}, D_*E_{2*}.

The quantities ρ_1 and ρ_2 are found implicitly from those equations, and the integrals in (3.3) in the x_1, x_2 plane cannot be evaluated exactly. It is better to change in (3.3) to variables of the hodograph plane u_1, u_2 for $t=t_*$.

The calculation of the Jacobian J (1.12) for $t=t_*$ leads to the representation

$$J_*=\frac{(\gamma+1)(3-\gamma)}{(\gamma-1)^2}(1+s_1\xi)^{\frac{2-\alpha}{\alpha-1}}(1+s_2\eta)^{\frac{2-\alpha}{\alpha-1}}(f_1+g_1\xi)(f_2+g_2\eta) \tag{3.4}$$

$$f_k=q_k+s_kp_k(\alpha-1)^{-1},\quad g_k=q_ks_k\alpha(\alpha-1)^{-1},\quad k=1,2$$

$$\xi=u_1,\quad \eta=u_1+\delta u_2$$

Then we obtain

$$M_2=2\rho_0\left(\frac{\gamma-1}{2}\right)^{\frac{2}{\gamma-1}}\int_0^{\infty}du_2\int_{hu_2}^{\infty}J_*\left(\frac{2}{\gamma-1}+hu_1+u_2\right)^{\frac{2}{\gamma-1}}du_1 \tag{3.5}$$

The quantity ρ_1 in (3.3) is independent of x_1. Thus, after changing to the integration variable u_2 using (2.3) we obtain

$$M_1=\rho_0\int_0^{\infty}\left(\frac{2}{\gamma-1}+u_2\right)^{\frac{2}{\gamma-1}}\left(1-\frac{u_2}{B_2}\right)^{\frac{1}{\alpha-1}}\times$$

$$\times(l_0+l_1u_2)\left[h_0+h_1\left(1-\frac{u_2}{B_2}\right)^{\frac{1}{\alpha-1}}+h_2\left(1-\frac{u_2}{B_2}\right)^{\frac{\alpha}{\alpha-1}}\right]du_2 \tag{3.6}$$

where the constants h_k, l_k depend on r_*, t_*, γ, α. The expressions for these constants are cumbersome and will not be given here. For any α and γ the integrals in (3.5) and (3.6) can be found numerically. Exact (very lengthy) analytic calculations have been carried out for $\alpha = 6/7$, $\gamma = 5/3$, $t_* = 1$, $r_* = 1/2$, when condition (2.2) is satisfied. It turns out that the mass balance (3.3) is not satisfied.† Analytic calculations with these parameter values show that

$$M_0 = \rho_0\sqrt{2}, \quad M_1 = \rho_0\sqrt{2}\frac{m_1}{n_1}, \quad M_2 = \rho_0\sqrt{2}\left(\frac{m_2}{n_2} + \frac{m_3}{n_3}\ln 3\right)$$

where m_k, n_k are integers. The presence of ln 3 in the expression for M_2 proves the imbalance. When $r_* = 1/2$ we obtain $2M_1 \approx 0.471$; $M_2 \approx 0.891$; $2M_1 + M_2 < M_0$. As r_* increases, the imbalance increases rapidly.

We have thus discovered that partial collapse occurs during non-self-similar compression of this type.

It is interesting to track the dynamics of point masses which lie on the bisector OS_*, all of which collapse to the point D_*. It turns out that this can be done exactly using Eq. (2.10), which describes the trajectories of particles moving along OS_*.

Suppose that at time $t = t_0$ a weak discontinuity P_{t0} has reached the point D_0 on the bisector, and the mass particle which is situated at the point D_0 at rest before $t \leqslant t_0$ starts to move. Its trajectory can be described by the equation (the analogue of (2.12))

$$-\tau = \left(1 + \frac{2}{\delta}b\right)^{-\frac{\gamma+1}{2(\gamma-1)}} G_0, \quad G_0 = t_0 + G \tag{3.7}$$

The greatest time t_0^*, when the trajectory of the corresponding point mass, which at some point D_{0*} was on the bisector at $t = t_0^*$, arrives at the point D_* at time t_*, is given by the equation

$$t_0^* = t_* - r\int_0^\infty R(x)\,dx \tag{3.8}$$

where $R(x)$ is the same as the integrand of (2.12). Representation (3.8) is obtained from the condition $G_0 \leqslant 0$.

Thus, all the points on the bisector OS_* to the left of the point D_0 collapse to the point D_*.

The description of the configuration of the entire initial sub-region of the prism with section $OM_*S_*N_*$, which contracts into segments D_*E_{1*} and D_*E_{2*}, can be obtained by numerical integration of the system of equations (2.9) using high-accuracy algorithms.

Thus, the energy expended in moving compressive pistons in the region of a double wave is infinite, although it is finite in regions of Riemann waves.

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†This fact was first discovered from numerical simulations by V. A. Kukushkin.

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